M-math 2nd year Mid Term
Subject : Stochastic Processes

Time : 3.00 hours Max.Marks 50.

1. a) Let $\{X_n, \mathcal{F}_n, n \geq 0\}$ be a sub-martingale. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex, non decreasing function. Show that $\{\phi(X_n), \mathcal{F}_n\}$ is a sub-martingale and prove your result.
b) Let $\{X_n, \mathcal{F}_n, n \geq 0\}$ be a sub-martingale. Show that if $X_n = M_n + A_n$ is the Doob decomposition of the process with increasing process (A_n), then the decomposition is unique if (A_n) is an $\mathcal{F}_n$-predictable sequence. (5+7)

2. Let $\{\Delta_n, n \geq 1\}$ be a sequence of integrable independent random variables with $E\Delta_n = 0$. Let $\{X_n, n \geq 1\}$ be defined as follows: $X_1 = \Delta_1$,
$X_{n+1} := X_n + \Delta_{n+1} f_n(X_1, \cdots, X_n), f_n : \mathbb{R}^n \to \mathbb{R}$, f_n bounded and measurable. Show that
$\{X_n, \mathcal{F}_n, n \geq 1\}$ is a martingale where $\mathcal{F}_n := \sigma\{\Delta_1, \cdots, \Delta_n\}, n \geq 1$. (10)

3. Let (B_t) be a standard one dimensional Brownian motion. For $K > 0$ define

$$\tau := \inf\{s > 0 : B_s > K\sqrt{s}\}.$$

Show that $P\{\tau = 0\} = 1$. (10)

4. Let (B_t) be as above.
a) Compute the mean and variance of $\int_0^1 B_s \, ds$.
b) Show that $\lambda\{t : B_t = 0\} = 0$, almost surely, where λ denotes Lebesgue measure. (8+7)

5. Let (B_t) be as above and $\sigma > 0$. Show that $\{B_t^2 - t; t \geq 0\}$ and $\{exp(\sigma B_t - \frac{\sigma^2}{2} t); t \geq 0\}$ are martingales. (10)